JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 16, No. 5, September–October 1993

Controllability and Observability of Flexible Structures with Proof-Mass Actuators

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Relationships between Hankel singular values of a structure with a proof-mass actuator to those of a structure with an ideal actuator are obtained for small damping and distinct natural frequencies. They indicate that if the natural frequency of the proof-mass actuator is much lower than the structural frequencies, the proof-mass actuator has little influence on Hankel singular values of a structure. This fact significantly simplifies controllability/observability analysis and model reduction of flexible structures under certain conditions. Results from numerical simulations verify the conclusions.

Introduction

MODEL order determination is an important step in structural dynamic analysis, identification procedures, and control design. A structural dynamic model obtained either from analysis (e.g., finite element) or from experiments typically needs to be reduced. Tools for model reduction[1-6] and order determination[7,8] of flexible structures have been developed using Hankel singular values (HSV) as a measure of a system's joint controllability and observability properties.[1] Although proof-mass actuators (PMAs) are widely used in structural dynamics testing, in most investigations actuator dynamics are not included in the model. An actuator and sensor in a cascade connection with a structure are analyzed in Refs. 6 and 9. Recognizing the importance of PMA dynamics in the control design problem, Zimmerman and Inman[10] and Zimmerman et al.[11] have investigated the interaction between structures and the PMAs and conclusively show that actuators must be tailored to fit a particular structure if improved performance is to be achieved.

In this paper the relationship between the HSV of a structure with PMA and the HSV of the structure alone (i.e., with an ideal actuator) is investigated. Using the new relationships derived herein, the influence of PMAs on model reduction is analyzed and demonstrated via simulations.

Two types of PMAs are considered. Both consist of mass m attached to a structure at node n_a. The first type, A PMA, is a reaction-type force actuator.[10-12] It generates a force by reacting against the proof mass m. An external force command f is applied equally and opposite onto the structure at node n_a and the proof mass (Fig. 1a). The remaining parameters d and k represent the inherent damping in the PMA system and the stiffness force to keep the proof mass centered. The reader is referred to Ref. 10 for a more detailed description of a type A actuator. The second type, B PMA, is a centrifugal actuator, where force is proportional to the square of the excitation frequency. The force acts on mass m exclusively (Fig. 1b). In this study, the practical limitation of PMA stroke length is not addressed. A recent study that uses a nonlinear control scheme to circumvent this limitation can be found in Ref. 13.

Structure with Type A Proof-Mass Actuator

Let us consider a structure without an actuator, shown in Fig. 1c. Its dynamics may be written as

$$M_s\ddot{q}_s + D_s\dot{q}_s + K_sq_s = B_sf; \qquad y = C_sq_s \tag{1}$$

where f is the force acting on the structure; q is its displacement vector ($n \times 1$); y is its output vector ($p \times 1$); M_s, D_s, and K_s are the mass, damping, and stiffness matrices of the structure, respectively; B_s is the matrix of actuator location,

$$B_s = [0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0]^T \tag{2}$$

with a nonzero term at location n_a; and C_s is a matrix of sensor location. If we denote $G_s(\omega) = -\omega^2M_s + j\omega D_s + K_s$, $j = \sqrt{-1}$,

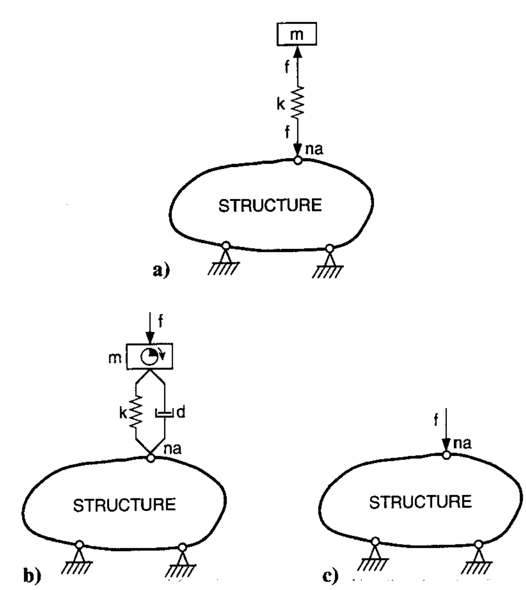

Fig. 1 Structure and actuator configuration.

Received Feb. 14, 1992; presented as Paper 92-2139 at the AIAA Dynamics Specialist Conference, Dallas, TX, April 16–17, 1992; revision received Oct. 23, 1992; accepted for publication Oct. 31, 1992.

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then the structure transfer function (the dynamic elasticity of the structure) is

$$H_s = C_s G_s^{-1} B_s \tag{3}$$

The dynamic stiffness of a structure at the actuator location is defined as

$$k_s = (B_s^T G_s^{-1} B_s)^{-1} \tag{4}$$

The dynamic stiffness is the inverse of the frequency response function, i.e., displacement to force relationship, at the actuator location. At zero frequency, it reduces to the stiffness constant at the actuator location. For a structure with a type A PMA, denote q_a, m, k, and d as the displacement, mass, stiffness, and damping of the actuator, respectively, and define the displacement vector $q_c^T = [q_s^T \quad q_a]$. The equation of combined structure and actuator is

$$M_c \ddot{q}_c + D_c \dot{q}_c + K_c q_c = B_c f; \qquad y = C_c q_c \tag{5}$$

where

$$M_c = \text{diag}(M_s, m) \qquad B_c^T = [B_s^T \quad -1]$$
$$C_c = [C_s \quad 0] \tag{6}$$

and

$$K_c = \begin{bmatrix} K_s + kB_sB_s^T & -kB_s \\ -kB_s^T & k \end{bmatrix} \tag{7}$$

$$D_c = \begin{bmatrix} D_s + dB_sB_s^T & -dB_s \\ -dB_s^T & d \end{bmatrix} \tag{8}$$

Denote $G_c(\omega) = -\omega^2 M_c + j\omega D_c + K_c$; then the transfer function of the structure with a type A PMA is

$$H_c = C_c G_c^{-1} B_c \tag{9}$$

Proposition 1: The relationship between the transfer function of the structure with a type A PMA and the transfer function of the structure alone is as follows:

$$H_c = \alpha_c H_s \tag{10}$$

$$\alpha_c = \begin{cases} \dfrac{1}{\hat{\beta} + 1 - \rho^2 - j2\zeta\rho} & \text{any damping} \\ \dfrac{1}{1 + \beta - \rho^2} & \text{small damping} \end{cases} \tag{11}$$

where $\rho = \omega_0/\omega$, $\omega_0 = \sqrt{k/m}$, $\hat{\beta} = \hat{k}/k_s$, $\beta = k/k_s$, $\hat{k} = k + j\omega d$, and $\zeta = d/2m\omega_0$.

Proof: Denote

$$H_0 = (G_s + \hat{k}B_sB_s^T)^{-1}, \qquad g = -\omega^2 m + \hat{k}$$
$$\Delta = g - \hat{k}^2 B_s^T H_0 B_s \tag{12}$$

then

$$G_c = \begin{bmatrix} -\omega^2 M_s + j\omega\hat{D}_s + \hat{K}_s & -\hat{k}B_s \\ -\hat{k}B_s^T & g \end{bmatrix}$$
$$= \begin{bmatrix} -H_0^{-1} & -\hat{k}B_s \\ -\hat{k}B_s^T & \Delta + \hat{k}^2 B_s^T H_0 B_s \end{bmatrix} \tag{13}$$

where $\hat{D}_s = D_s + dB_sB_s^T$, and $\hat{K}_s = K_s + kB_sB_s^T$. By applying the partitioned inverse formula[14] to the previous equation, it can be shown that

$$G_c^{-1} = \Delta^{-1} \begin{bmatrix} \Delta H_0 + \hat{k}^2 H_0 B_s B_s^T H_0 & -\hat{k}H_0B_s \\ -\hat{k}H_0B_s^T & 1 \end{bmatrix} \tag{14}$$

Next, from Eqs. (10) and (14) one obtains

$$H_c = C_s\left[(H_0 + \hat{k}^2\Delta^{-1}H_0B_sB_s^TH_0)B_s - \Delta^{-1}\hat{k}H_0B_s\right]$$
$$= C_s\left[I - \hat{k}\Delta^{-1}(I - \hat{k}H_0B_sB_s^T)\right]H_0B_s \tag{15}$$

and from Eq. (12)

$$H_0 = \left[I - \frac{\hat{k}G_s^{-1}B_sB_s^T}{(1 + \hat{k}B_s^TG_s^{-1}B_s)}\right]G_s^{-1} \tag{16}$$

Thus,

$$H_0B_s = (1 + \hat{\beta})^{-1}G_s^{-1}B_s \tag{17}$$

Combining Eqs. (9), (14), and (17) gives Eq. (10). □

Corollary 1: For the case where

$$\omega \gg \omega_0 \ (\rho \ll 1), \qquad k \ll k_s \ (\beta \ll 1), \qquad \zeta \ll 1 \tag{18}$$

are satisfied, one obtains $\alpha_c \cong 1$. In this case the transfer function of the system with a PMA is approximately equal to the transfer function of the system without a PMA.

For a PMA with a soft centering spring with low friction, the conditions given by Eq. (18) are approximately satisfied.

Proposition 2: For small damping and distinct natural frequencies, the Hankel singular values of a structure alone, γ_{si}, and of a structure with a type A PMA, γ_{ci}, are related as follows:

$$\gamma_{si} = \gamma_{ci}/\alpha_{ci}; \qquad i = 1, \ldots, 2n \tag{19}$$

where

$$\alpha_{ci} = \alpha_c(\omega_i) = 1/(1 + \beta_i - \rho_i^2) \tag{20}$$

$$\rho_i = \rho(\omega_i) = \omega_0/\omega_i \tag{21}$$

$$\beta_i = \beta(\omega_i) = k/k_{si} \tag{22}$$

$$k_{si} = k_s(\omega_i) = \left[B_s^TG_s^{-1}(\omega_i)B_s\right]^{-1} \tag{23}$$

The variable k_{si} is the ith modal stiffness of the structure.

Proof: For small damping and distinct natural frequencies, the HSV of a flexible structure are[5,6]

$$\gamma_{si} \cong 0.5\|H_s(\omega_i)\|; \qquad i = 1, \ldots, 2n \tag{24}$$

where $\|A\| = \text{tr}(AA^*)$ is a spectral norm of A, A^* is a complex conjugate transpose of A, and ω_i is the ith natural frequency of the structure. Introducing Eqs. (10–24), one obtains Eq. (19). □

In addition to the conditions stated in Proposition 2, consider the following conditions:

$$\omega_0 \ll \omega_1, \quad \text{and} \quad k \ll \min_i k_{si} \tag{25}$$

where ω_1 is the fundamental (lowest) frequency of the structure. The conditions say that the actuator natural frequency should be much smaller than the fundamental frequency of the structure, and the actuator stiffness should be much smaller than the dynamic stiffness of the structure at any frequency of interest. Drawing from single degree-of-freedom (SDOF) modal analysis curve-fitting theory (see for example, Ref. 15), if a given structural mode is far from another struc-

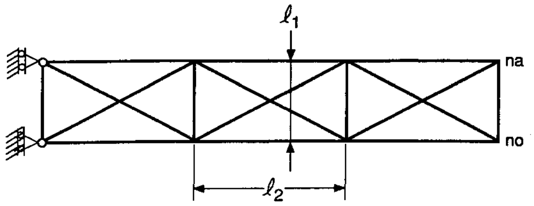

Fig. 2 Truss structure.

tural mode, then one can assume the model to be an SDOF oscillator in that frequency range. In addition, for widely spaced modes, the "tail" of the frequency response of that mode on all other modes can be ignored (essentially, the tail goes to zero as frequency is increased). Thus, if the PMA mode is far from any structural mode, the structure will exhibit uncoupled (to the PMA) frequency response at the higher structural frequencies.

If the aforementioned conditions are satisfied, one obtains $\alpha_{ci} \cong 1$ for $i = 1, \ldots, n$; thus the HSV of the structure with a PMA are equal to the HSV singular values of the structure without a PMA. The controllability and observability properties of the system are preserved so that the presence of the PMA does not influence the controllability and observability of the structure. In particular, the presence of the PMA will not affect the model order reduction. Note also that, for many cases, whenever the first condition of Eq. (25) is satisfied, the second condition is satisfied too.

Example 1: A truss structure from Fig. 2 is investigated for $l_1 = 70$ in. and $l_2 = 100$ in. Each truss has a cross-sectional area of 2 in.^{2}, elastic modulus of 10^6 lb/in.^{2}, and mass density of 2 lb-s^2/in.^{2} A vertical force is applied at node n_a, and the output is measured at node n_0 in the vertical direction. The magnitude of the transfer function of this structure is presented in Fig. 3 (solid line). The PMA is applied at node n_a, with mass $m = 200$ lb-s^2/in. and stiffness $k = 0.6$ lb/in.; hence its natural frequency is $\omega_0 = 0.0548$ rad/s. The actuator amplification factor α_c is plotted in Fig. 4. The figure indicates that for $\omega \gg \omega_0$ the amplification is equal to 1. The plot of the magnitude of the transfer function for the structure with the PMA is shown in Fig. 3 as a dashed line. The figure shows perfect overlapping of the transfer functions for $\omega \gg \omega_0$. HSV of the structure without and with a PMA are plotted in Fig. 5 (solid and dashed line, respectively). Observe that the HSV are the same in both cases, except for the first one, which is related to the PMA itself.

Structure with Type B Proof-Mass Actuator

This configuration is shown in Fig. 1b. The force acting on mass m is proportional to squared frequency

$$f = \kappa\omega^2 u \tag{26}$$

where κ is a constant, and u is the input to the type B PMA.

Proposition 3: The relationship between transfer functions of a structure with and without type B PMA is as follows:

$$H_c = \alpha_c H_s \tag{27}$$

$$\alpha_c = \begin{cases} \dfrac{\kappa\omega_0^2(1 + j2\zeta/\rho)}{\beta + 1 - \rho^2 - j2\zeta\rho} & \text{any damping} \\ \dfrac{\kappa\omega_0^2}{1 + \beta - \rho^2} & \text{small damping} \end{cases} \tag{28}$$

Proof: For the displacement vector $q_c^T = [q_s^T \ q_a]$; the matrices M_c, D_c, K_c, and C_c as in Eqs. (6) and (8); and $B_c^T = [0 \ \ \kappa\omega^2]$; one obtains G_c^{-1} as in Eq. (14) and consequently Eq. (27). □

Corollary 2: For the conditions in Eq. (25), one obtains the constant α_c,

$$\alpha_c = \kappa\omega_0^2 \tag{29}$$

The previous result shows that the structural transfer function with a type B PMA is proportional to the structural transfer function without a PMA.

Proposition 4: For small damping and distinct natural frequencies, the HSV of a structure alone, γ_{si}, and a structure with a type B PMA, γ_{ci}, are related as in Eq. (19), with

$$\alpha_{ci} = \frac{\kappa\omega_0^2}{1 + \beta_i - \rho_i^2} \tag{30}$$

With the additional conditions in Eq. (25), one obtains $\alpha_{ci} = \kappa\omega_0^2$ for $i = 1, \ldots, n$; thus the HSV of the structure with a type B PMA are proportional to the HSV of the structure without a PMA. This scaling does not influence the results of reduction

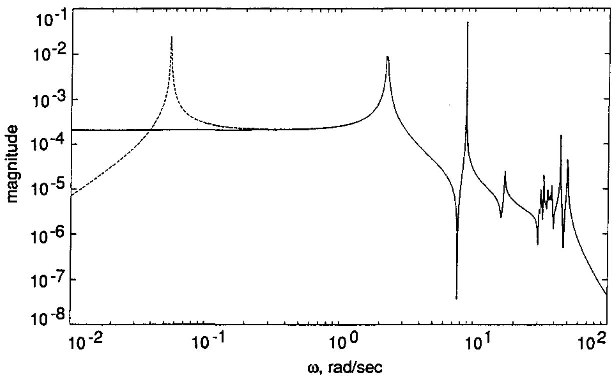

Fig. 3 Transfer function of the truss structure without PMA (solid line) and with PMA (dashed line).

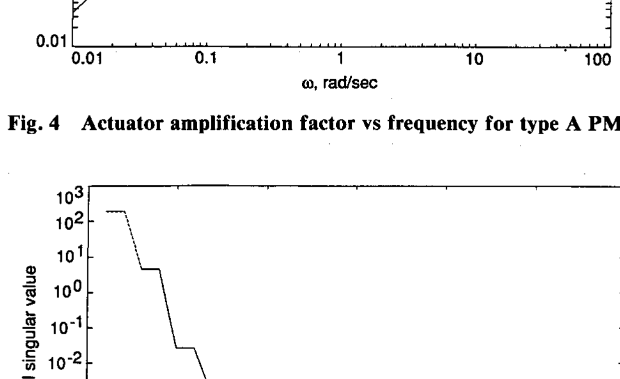

Fig. 4 Actuator amplification factor vs frequency for type A PMA.

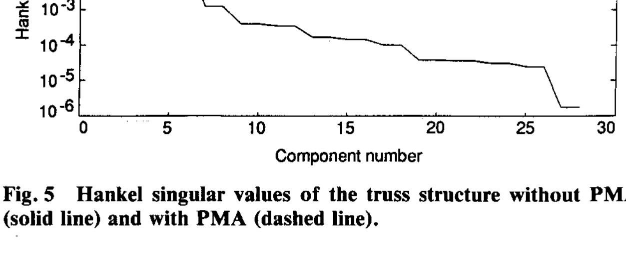

Fig. 5 Hankel singular values of the truss structure without PMA (solid line) and with PMA (dashed line).

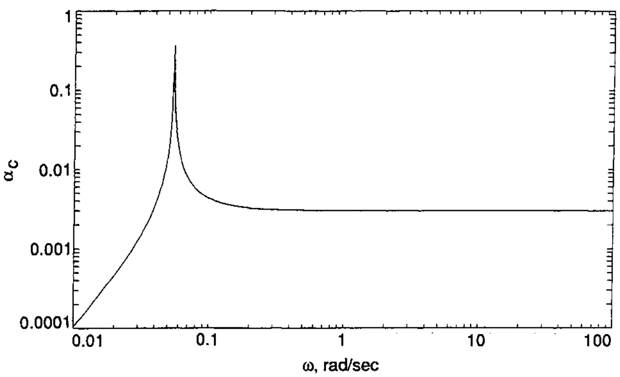

Fig. 6 Actuator amplification factor vs frequency for type B PMA.

or model order determination procedures since the procedures are based on HSV ratios rather than their absolute values.

In applications where measurement noise is unavoidable, care should be taken in the selection of the scaling factor. For instance, if the scaling factor is too small, this may produce small HSV that cannot be detected, and the reduction or order determination procedure could then be biased.

Example 2: The system described in Example 1 is investigated. Amplification factor α_c as defined in Eq. (30) is plotted in Fig. 6. It is constant and equal to $\omega_0^2=0.003$ for $\omega \gg \omega_0$. Thus, to obtain the transfer function of the structure, the transfer function of the structure with a PMA is divided by ω_0^2. Plots of magnitudes of the transfer functions of the structure (solid line) and the structure with a PMA divided by ω_0^2 (dashed line) are exactly the same as in Fig. 3. Both plots overlap for $\omega \gg \omega_0$. Similarly, plots of HSV of the structure and HSV of the structure divided by ω_0^2 are the same as in Fig. 5 and overlap for $\omega \gg \omega_0$.

Conclusions

A simple expression has been derived that relates the transfer function of the structure with an attached PMA and the structure only. For small damping and distinct natural frequencies, relationships between the HSV for the aforementioned configurations are obtained. Furthermore, it has been shown that under mild conditions on the type A PMA (where the natural frequency of the PMA is much smaller than the structure's fundamental frequency), the HSV remain unchanged. Thus, any model reduction procedure based on HSV will be unaffected by the presence of a type A PMA. For a structure with a type B PMA, the HSV are scaled by a constant. The scaling does not influence the model order reduction procedures either since the procedure is based on ratios of HSV rather than their absolute values. However, when scaling produces very small singular values, care should be taken to avoid numerical difficulties.

Acknowledgments

The work described in this paper was conducted in part at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with NASA. The authors would like to thank the reviewers for their helpful comments.

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